YAU COLLEGE MATH CONTESTS INDIVIDUAL ALGEBRA 2018

Problem 1

Factorize the polynomial

$$f(x) = 6x^5 + 3x^4 - 9x^3 + 15x^2 - 13x - 2$$

into a product of irreducible polynomials in the ring $\mathbb{Q}[x]$.

Problem 2

Prove that any group of order 588 is solvable, given that any group of order 12 is solvable.

Problem 3

Decide which field F has the following property: for each integer $n > 0$, and for every $n \times n$ matrix A with entries in F, we can conjugate A to an upper triangular matrix under $GL_n(F)$.

Problem 4

Let n be a positive integer.

(1) Find the image of the map

$$M_{n\times n}(\mathbb{C}) \longrightarrow M_{n\times n}(\mathbb{C}), \quad A \to A^t A.$$

Here $M_{n\times n}(\mathbb{C})$ denotes the space of all $(n \times n)$ matrices with complex entries.

(2) Find the image of the map

$$M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R}), \quad A \to A^t A.$$

Here $M_{n\times n}(\mathbb{R})$ denotes the space of all $(n \times n)$ matrices with real entries.

Problem 5

Let k be a field of characteristic $p > 0$, and let x, y be algebraically independent over k. Prove the following

(a). $k(x, y)$ has field extension degree p^2 over $k(x^p, y^p)$.

(b). There are infinitely many intermediate field extensions between $k(x, y)$ and $k(x^p, y^p)$.